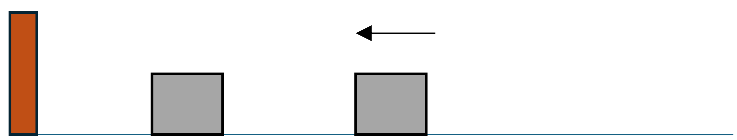
Teacher notes
Topic A

A shocking and beautiful result!

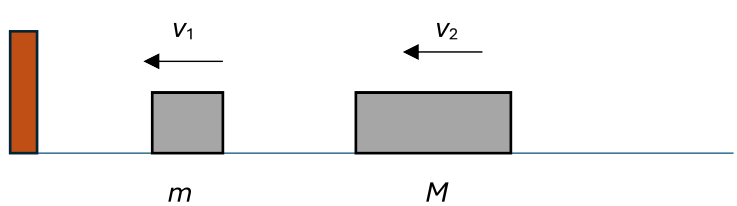
A block is at rest on a frictionless floor. An identical block moving towards the left collides elastically with the stationary block. How many collisions will there be between the blocks or a block and the wall on the left? The wall collisions are also elastic.



It is easy to see that in the first collision, the block on the right will come to rest and the left block will move to the left colliding with the wall. It will then bounce back and collide with the block to the right for the third and final collision.

How many such collisions will there be when the block on the right has a mass that is $M = 100^{n-1}m$ where *m* is the mass of the block on the left? For *n* = 1 we saw that the answer is 3. What is the answer in general? The answer is as shocking as it is beautiful! It was discovered by Gregory Galperin in the late 1990's.

Because the collisions are elastic kinetic energy is conserved. If at some point the velocities of the two blocks are v_1 and v_2



then

$$\frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 = \text{constant.}$$

Calling $y = v_1\sqrt{m_1}$ and $x = v_2\sqrt{M}$ the energy condition is simply $x^2 + y^2 = \text{constant}$ which is the equation of a circle. At all times the motion is represented by a point on this circle. Without loss of generality, we may take the value of the constant to be 1.

Momentum conservation gives

$$mv_1 + Mv_2 = \text{constant}$$

In terms of the new variables this becomes

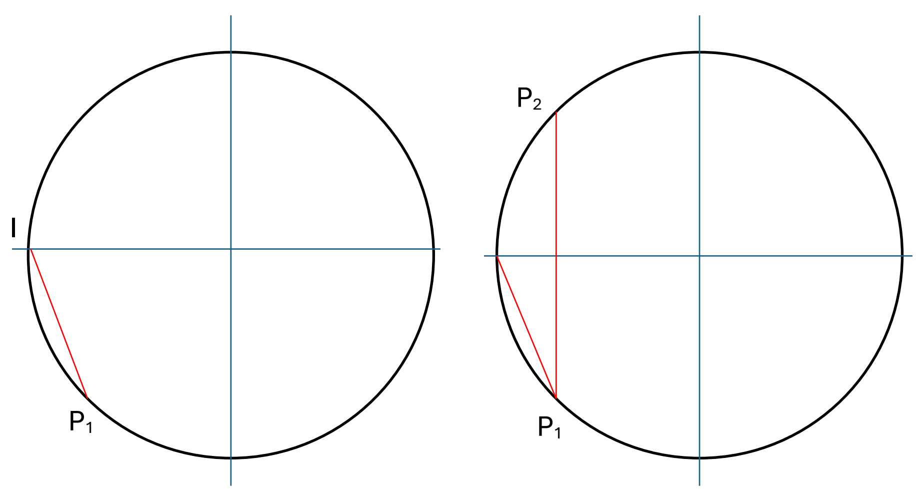
$$x\sqrt{M} + y\sqrt{m} = \text{constant}$$

Initially, $x = -1$ and $y = 0$ so the last constant is $-\sqrt{M}$, i.e. momentum conservation demands:

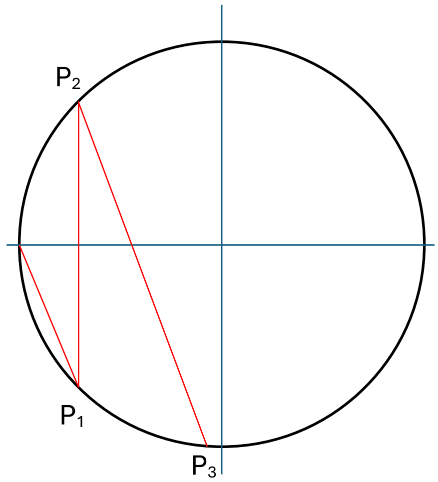
$$x\sqrt{M} + y\sqrt{m} = -\sqrt{M}$$

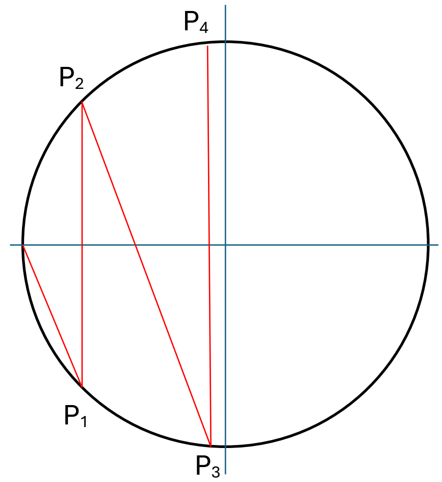
which is a straight line with gradient $-\sqrt{\frac{M}{m}} = -\sqrt{100^{n-1}} = -10^{n-1}$.

The initial state of our system is represented by point I in the left diagram below: the heavy block is moving towards the stationary light block. The first collision occurs at P_1 and the velocities of both blocks change. The light block acquires a negative velocity and moves towards the wall. The collision with the wall is elastic and so the light block changes direction at the same speed, point P_2. This is the second collision.

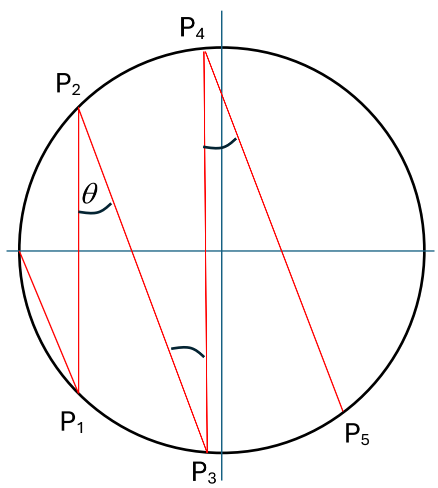


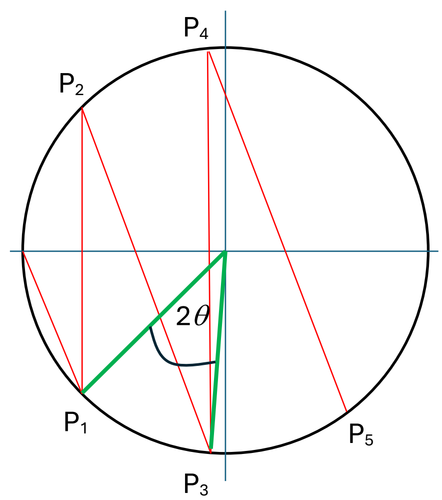
The third collision occurs at P_3. Momentum conservation demands that we are on the same parallel slanted line beginning at P_2.





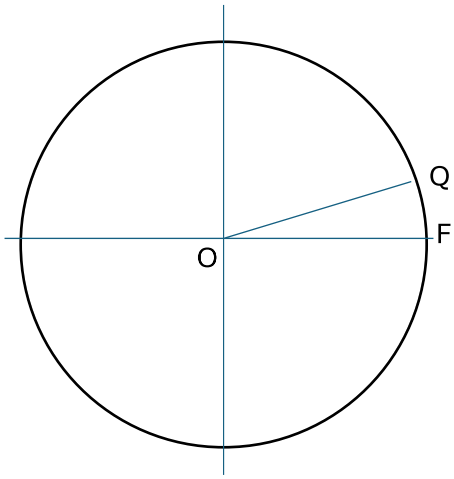
Points P divide the circumference of the circle into arcs. The marked angles are all equal and equal to θ. The angles subtended **at the center of the circle** by each of the arcs is then 2θ. Hence, **all** arcs have the same length and that length is 2θ since we took the circle to have radius 1.





The number of collisions will be the number of points P on the circumference which is equivalent to the number of arcs the circumference is divided into.

How many points P will there be? The collisions will stop when the heavy block has a positive velocity greater than the velocity of the light block. This means that the heavy block will move to infinity on the right and the light block will never again catch up to collide with it. The line OQ represents the state when both blocks have the same velocity. We need the last point P to end up in between Q and F.



This means that that the number of collisions is the largest integer N such that:

$$N(2\theta) < 2\pi \quad \text{or} \quad N\theta < \pi$$

Remember that the slanted straight lines have gradient -10^{n-1}. Hence $\theta = \arctan\frac{1}{10^{n-1}}$. From

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

it is an excellent approximation that $\arctan x \approx x$. Hence

$$\theta = \arctan\frac{1}{10^{n-1}} \approx \frac{1}{10^{n-1}}$$

n = 2: $\theta \approx 0.1$. Then $N \times 0.1 < \pi$ so N = 31

n = 3: $\theta \approx 0.01$. Then $N \times 0.01 < \pi$ so N = 314

n = 4: $\theta \approx 0.001$. Then $N \times 0.001 < \pi$ so N = 3141

n = 5: $\theta \approx 0.0001$. Then $N \times 0.0001 < \pi$ so N = 31415

We see the extraordinary result that the number of collisions is given by the digits of π!

You can find more details and beautiful graphics in the great series of 3Blue1Brown at

https://www.youtube.com/watch?v=jsYwFizhncE&t=635s